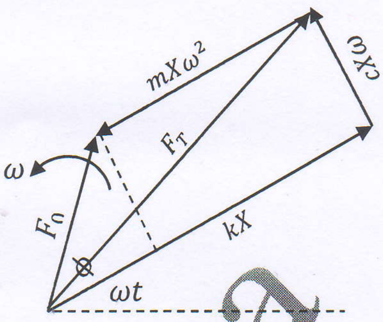
$$X = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \frac{100/355753}{\sqrt{[1-(1.8746)^2]^2+[2\times0.1\times1.8746]^2}}$$



$$X = \frac{0.00028109}{2.54192} = 110.6\times10^{-6}\ \text{m}$$

the transmitted force $= F_T = \sqrt{[kX]^2+[cX\omega]^2}$

$$F_T = kX\sqrt{1+\left[\frac{c\omega}{k}\right]^2} = kX\sqrt{1+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

$$F_T = 355753\times110.6\times10^{-6}\sqrt{1+[2\times0.1\times1.87462]^2} = 42.0\ N$$

There is considerable interest in the case of light damping, such as when $\zeta < 0{,}05$, in which case peaks occur in the immediate neighborhood of $\omega/\omega_n = 1$. Moreover, for small values of ζ Equation (3) yields the approximation

$$\left.\frac{Xk}{F_0}\right|_{max} = \text{Q} = \frac{1}{2\zeta}$$ Q is **quality factor** ………… (4)

Equation (4) can be used as a quick way of estimating the viscous damping factor ζ of a system by producing the plot $|G(i\omega)| = Xk/F_0$ versus ω/ω_n experimentally, measuring the peak amplitude Q and writing

$$\zeta = \frac{1}{2\text{Q}} \qquad \text{…………} (5)$$

The points P_1 and P_2, where the amplitude of $|G(i\omega)|$ falls to $\text{Q}/\sqrt{2}$, are called **half-power points**, because the power absorbed by the resistor in an electric circuit or by the damper in a mechanical system subjected to a harmonic force is proportional to the square of the amplitude. To obtain the driving frequencies corresponding to P_1 and P_2, we use Equations (2) and (4) and write

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \frac{1}{\sqrt{2}}\text{Q} = \frac{1}{2\sqrt{2}\,\zeta} \qquad \text{…………} (6)$$

which yield

$$8\zeta^2 = \left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2 \quad \text{or} \quad 8\zeta^2 = 1-2\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2$$

$$\left(\frac{\omega}{\omega_n}\right)^4 - 2(1-2\zeta^2)\left(\frac{\omega}{\omega_n}\right)^2 + 1 - 8\zeta^2 = 0$$ quadratic Equation in $(\omega/\omega_n)^2$

$$\begin{matrix}(\omega_1/\omega_n)^2\\(\omega_2/\omega_n)^2\end{matrix} = (1-2\zeta^2) \mp \sqrt{(1-2\zeta^2)^2-(1-8\zeta^2)} \cong 1-2\zeta^2 \mp 2\zeta$$

or

$(\omega_1/\omega_n)^2 \cong 1 - 2\,\zeta^2 - 2\zeta \quad \cdots\cdots (a) \qquad (\omega_2/\omega_n)^2 \cong 1 - 2\,\zeta^2 + 2\zeta \quad \cdots\cdots (b)$

where ω_1 and $\omega_2$are the excitation frequencies corresponding to P_1 and P_2, respectively

subtracting (b) from (a)

$$\left(\frac{\omega_2}{\omega_n}\right)^2 - \left(\frac{\omega_1}{\omega_n}\right)^2 = 4\zeta \qquad \cdots\cdots\cdots\cdots (6)$$

$$\omega_2^2 - \omega_1^2 = \omega_n^2 4\zeta \qquad \Longrightarrow \qquad (\omega_2 - \omega_1)(\omega_2 + \omega_1) = \omega_n^2 4\zeta$$

Then recognizing that for light damping $\omega_1 + \omega_2 \cong 2\omega_n$ $\qquad \omega_2 - \omega_1 = \Delta\omega$

$$\Delta\omega \times 2\omega_n = \omega_n^2 4\zeta \qquad \Delta\omega \cong 2\zeta\omega_n \qquad \cdots\cdots\cdots\cdots (7)$$

The increment of frequency $\Delta\omega$ associated with the half-power points P_1 and P_2 is referred to as the *bandwidth* of the system. Inserting Equation (7) into Equation (4), we obtain

$$Q \cong \frac{1}{2\,\zeta} \cong \frac{\omega_n}{\Delta\omega} = \frac{\omega_n}{\omega_2 - \omega_1} \qquad \cdots\cdots\cdots\cdots ()$$

so that the requirement of high quality factor is equivalent to small bandwidth.

Example

A mass-damper-spring system has been observed to achieve a peak magnification factor Q = 5 at the driving frequency $\omega = 10\ rad/sec$. It is required to determine: (1) the damping factor, (2) the driving frequencies corresponding to the half-power points and (3) the bandwidth of the system.

Solution

$$\zeta = \frac{1}{2Q} = \frac{1}{2\times 5} = 0.1$$

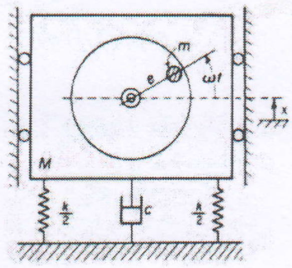
driving frequencies ω_1 and ω_2 can be determined from equations (a) and (b) where $\omega_n = 10\ rad/sec$

$$\omega_1 = \sqrt{1 - 2\,\zeta^2 - 2\zeta}\ \omega_n = 8.832\,rad/sec$$

$$\omega_2 = \sqrt{1 - 2\,\zeta^2 + 2\zeta}\ \omega_n = 10.86\,rad/sec$$

the bandwidth of the system $\Delta\omega = \omega_2 - \omega_1 = 2.028\ rad/sec$

Rotating Un-balance



Let

M: – total system mass

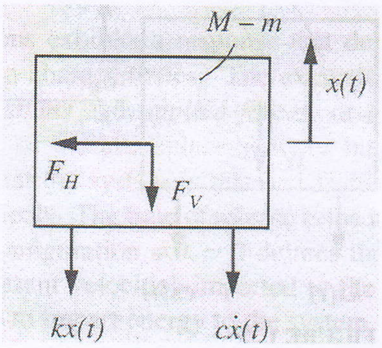
m: – eccentric mass

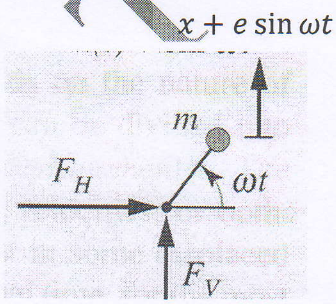
e: – eccentricity

ω: – angular velocity of eccentric mass

$x(t)$: – linear displacement of the non-rotating mass $(M - m)$

To drive the equation of motion, it is convenient to consider two free-body diagrams, one for $M - m$ and one for m.





where

for $M - m$ applying Newton's second law

$\sum F_y = (M - m)\ddot{y}$ $-F_V - kx - c\dot{x} = (M - m)\ddot{x}$

From the figure the displacement of eccentric mass $(m) = x + e \sin \omega t$

for m applying Newton's second law

$F_V = m\frac{d^2}{dt^2}(x + e \sin \omega t) = m(\ddot{x} - e\omega^2 \sin \omega t)$

eliminating the vertical reaction F_V and rearranging

$$\boxed{M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t}$$ **(equation of motion)** ⋯⋯⋯⋯ (1)

From homogeneous part $\omega_n^2 = \frac{k}{M}$ and $\frac{c}{M} = 2\zeta\omega_n$

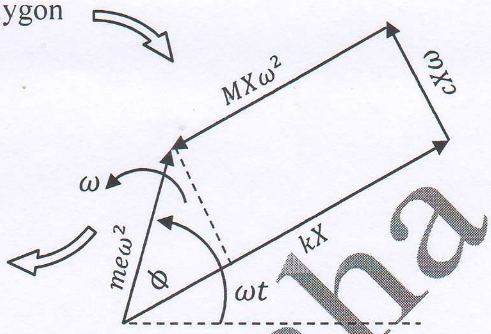
Where $me\omega^2 \sin \omega t$ is excitation force with amplitude $me\omega^2$

Now let $x(t) = X\sin(\omega t - \phi)$ = steady-state response ϕ:- is phase angle between response and excitation

Substituting (t) , $\dot{x}(t)$ and $\ddot{x}(t)$ into equation (1)

$$-MX\omega^2 \sin(\omega t - \phi) + cX\omega \sin(\omega t - \phi + \tfrac{\pi}{2}) + kX \sin(\omega t - \phi) = me\omega^2 \sin \omega t$$

From this equation we can graph the following force polygon



then from this polygon we can extract that

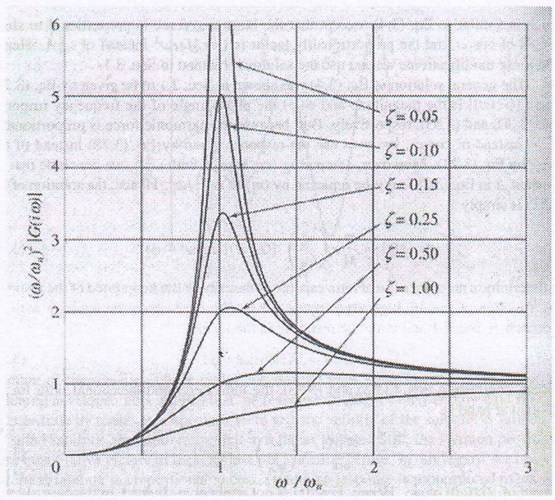
$$X = \frac{me\omega^2}{\sqrt{[k-M\omega^2]^2+[c\omega]^2}} \times \frac{M/k}{M/k}$$

since $\omega_n^2 = \frac{k}{M}$ and $\frac{c\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$

so $\frac{MX}{me} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$ and $\tan\phi = \frac{c\omega}{k-M\omega^2} \times \frac{k}{k} = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$

$$\boxed{\phi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}}$$

to sketch the amplitude ratio $\frac{MX}{me}$ with $\frac{\omega}{\omega_n}$ for $\zeta = 0.4$



let $\frac{\omega}{\omega_n} = 0$ $\frac{MX}{me} = 0$ for any value of ζ

and $\frac{\omega}{\omega_n} = 1$ $\frac{MX}{me} = \frac{1}{2\zeta} = \frac{1}{0.8} = 1.25$

to find the maximum value $\frac{d}{d\omega}\left(\frac{MX}{me}\right) = 0$

this yield $\frac{\omega}{\omega_n} = \frac{1}{\sqrt{1-\zeta^2}} = \frac{1}{\sqrt{1-(0.4)^2}} = 1.091$

to the right of $\frac{\omega}{\omega_n} = 1$

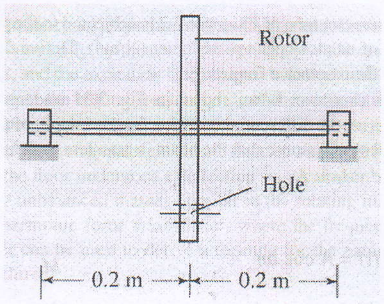
$\frac{\omega}{\omega_n} \to \infty$ $\frac{MX}{me} \to 1$ for any value of ζ

HWs

1- A piece of machinery can be regarded as a rigid mass with two reciprocating rotating unbalanced masses. The total mass of the system is 12 kg and each of the unbalanced masses is equal to 0.5 kg. during normal operation, the rotation of the masses varies from

0 to 600 rpm. Design a support system so that the maximum vibration amplitude will not exceed 10 percent of the rotating masses' eccentricity.

2- The rotor of a turbine having the form of a disk is mounted at the midspan of a uniform steel shaft as shown in Figure. The mass of the disk is 15 kg and its diameter is 0.3 m. the disk has a circular hole of diameter 0.03 m at a distance of 0.12 m from the geometric center. The bending stiffness of the shaft is = $1600\ \text{N.m}^2$. Determine the amplitude of vibration if the turbine rotor rotates with the angular velocity of 6000 rpm. Assume that the shaft bearings are rigid.



Solution $\quad M = 15\ \text{kg} \qquad EI = 1600\ \text{N.m}^2$

$$V = V_{disk} - V_{hole} = \left(\frac{0.3}{2}\right)^2 \pi \times \underbrace{1}_{\substack{thickness\\ of\ disk}} - \left(\frac{0.03}{2}\right)^2 \pi \times \underbrace{1}_{\substack{thickness\\ of\ disk}} = 0.07065 - 7.065 \times 10^{-4}$$

$$V = 0.069944\ m^3$$

$$\rho = \text{density} = \frac{M}{V} = \frac{15}{0.069944} = 214.46\ \text{kg}/m^3$$

so $\quad m = 214.46 \times 7.065 \times 10^{-4} = 0.152\ \text{kg}$

since the stiffness of Simple Supported Beam (pp-8) is

$$k = \frac{48\ EI}{l^3} = \frac{48 \times 1600}{(0.2 + 0.2)^3} = 1200\ \frac{kN}{m}$$

since $\quad \omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{1200\times10^3}{15}} = 282\ rad/sec \quad , \quad \omega = 6000 \times \frac{2\pi}{60} = 628\ rad/sec$

$$\frac{\omega}{\omega_n} = \frac{628}{282} = 2.22\ , \qquad \left(\frac{\omega}{\omega_n}\right)^2 = 4.928$$

from amplitude ratio for $\zeta = 0$

$$\frac{MX}{me} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \quad \text{or} \quad \frac{15X}{0.152 \times 0.12} = \frac{4.928}{1 - 4.928}$$

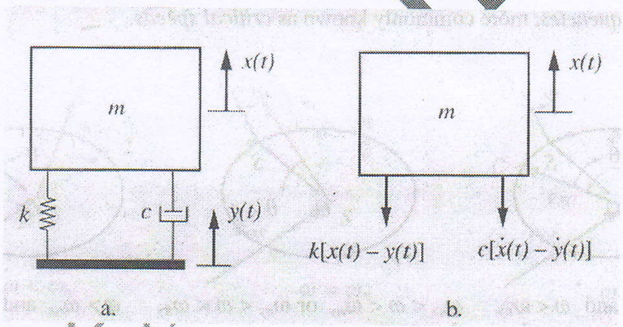
$$X = 0.001526\ m \quad \text{or} \quad 1.526\ mm$$

Harmonic Motion Of The Base

On occasions, sensitive equipment must be placed on a foundation undergoing undesirable vibration. To protect the equipment, it is necessary to isolate it from the damaging effects of the vibrating foundation. This can be achieved through rubber mounts acting both as springs and dampers.

Now, let $y(t):-$ be the base displacement and $x(t):-$ be the mass displacement

From the Figure (3) let $x(t) > y(t)$, so the free body diagram become to



Applying Newton's second law

$$-k(x-y)-c(\dot{x}-\dot{y})=m\ddot{x}$$

After re arranging $m\ddot{x}+c\dot{x}+kx=c\dot{y}+ky$ ………… (1)

From homogeneous part $\omega_n^2=\frac{k}{m}$ and $\frac{c}{m}=2\zeta\omega_n$, $\frac{c\omega}{k}=2\zeta\frac{\omega}{\omega_n}$

since $y(t)=Y\sin\omega t$ $\dot{y}(t)=Y\omega\cos\omega t=Y\omega\sin(\omega t+\frac{\pi}{2})$

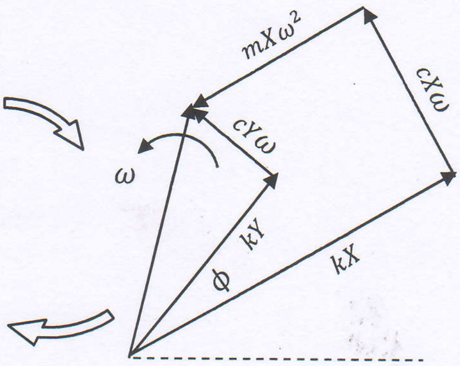
Now let $x(t)=X\sin(\omega t-\phi)$ = steady-state response ϕ:- is phase angle between response and excitation

Substituting (t) , $\dot{x}(t)$ and $\ddot{x}(t)$ into equation (1)

$$-mX\omega^2\sin(\omega t-\phi)+cX\omega\sin(\omega t-\phi+\frac{\pi}{2})+kX\sin(\omega t-\phi)=kY\sin\omega t+cY\omega\sin(\omega t+\frac{\pi}{2})$$

From this equation we can graph the following force polygon

then from this polygon we can extract that



$$(cY\omega)^2+(kY)^2=(kX-m\omega^2X)^2+(cX\omega)^2$$

$$\frac{X}{Y}=\sqrt{\frac{k^2+(c\omega)^2}{[k-m\omega^2]^2+[c\omega]^2}}\times\frac{k}{k}$$

since $\omega_n^2 = \frac{k}{m}$, $\frac{c}{m} = 2\zeta\omega_n$ and $\frac{c\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$

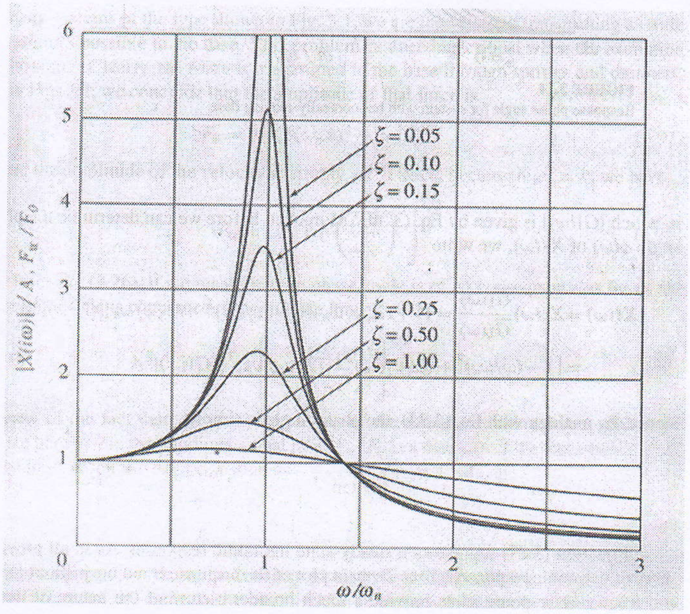
so $\frac{X}{Y} = \sqrt{\frac{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$ and $\phi = \alpha - \beta$ $\tan\phi = \tan(\alpha - \beta)$

since $\tan\phi = \frac{\tan\alpha - \tan\beta}{1-\tan\alpha\tan\beta}$

$$\tan\alpha = \frac{c\omega}{k-m\omega^2}\times\frac{k}{k} \qquad \tan\beta = \frac{c\omega}{k}$$

$$\tan\alpha = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad \tan\beta = 2\zeta\frac{\omega}{\omega_n}$$

So $\tan\phi = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)^3}{1-\left(\frac{\omega}{\omega_n}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$



to sketch the amplitude ratio $\frac{X}{Y}$ with $\frac{\omega}{\omega_n}$ for $\zeta = 0.4$

let $\frac{\omega}{\omega_n} = 0$ $\frac{X}{Y} = 1$ for any value of ζ

and $\frac{\omega}{\omega_n} = 1$ $\frac{X}{Y} = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta} = 1.6$

when $\frac{\omega}{\omega_n} = \sqrt{2}$ $\frac{X}{Y} = 1$ for any value of ζ

to find the maximum value $\frac{d}{d\omega}\left(\frac{X}{Y}\right)^2 = 0$

this yield $\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$ to the left of $\frac{\omega}{\omega_n} = 1$

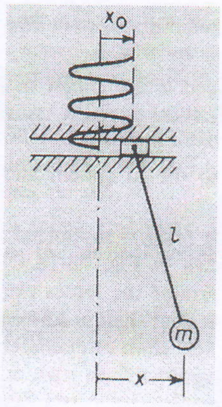
so for $\zeta = 0.4$ $\frac{\omega}{\omega_n} = 0.8246$

$$\left.\frac{X}{Y}\right|_{max} = \left.\frac{X}{Y}\right|_{\omega/\omega_n=0.8246} = 1.634$$

$\frac{\omega}{\omega_n} \to \infty$ $\frac{X}{Y} \to 0$ for any value of ζ

Example

The point of suspension of a simple pendulum is given by a harmonic motion $x(t) = X_o \sin \omega t$ a long a horizontal line. Write the differential equation of motion for a small amplitude of oscillation using the coordinates shown. Determine the solution for X/X_o, and show that when $\frac{\omega}{\omega_n} = \sqrt{2}$ the node is found at the midpoint of l. Show that in general the distance h from the mass to node is given by the relation $h = l\left(\frac{\omega_n}{\omega}\right)^2$ where $\omega_n = \sqrt{\frac{g}{l}}$



Solution

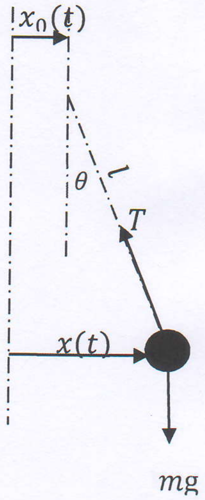
$$\sum F = m\,\ddot{x}$$

$$-T \sin\theta = m\,\ddot{x}$$

For small angle $T = mg$

$$-mg\theta = m\,\ddot{x} \qquad \cdots\cdots\cdots (a)$$

but $\quad \sin\theta = \frac{x - x_0}{l}$



then equation (a) becomes to

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}x_0 \qquad \textbf{equation of motion} \qquad \cdots\cdots\cdots (b)$$

where $\qquad \omega_n = \sqrt{\frac{g}{l}}$

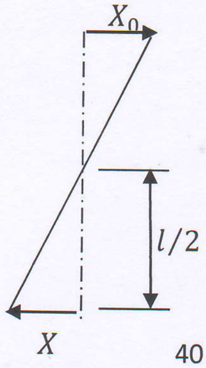
now, let $x_0(t) = X_0 \sin \omega t$ then $x(t) = X \sin \omega t \quad \Rightarrow \quad \ddot{x}(t) = -\omega^2 X \sin \omega t$

substituting into equation (b) $\qquad = -\omega^2 X \sin \omega t + \omega_n^2 X \sin \omega t = \omega_n^2 X_0 \sin \omega t$

$$X(\omega_n^2 - \omega^2) = \omega_n^2 X_0 \qquad X = \frac{X_0}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad \cdots\cdots\cdots (c)$$

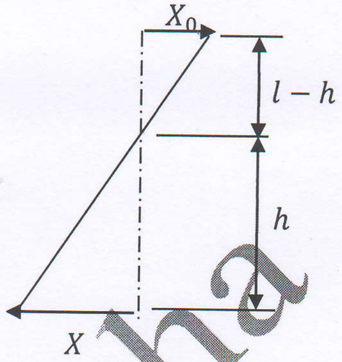
now, if $\omega / \omega_n = \sqrt{2}$

then from equation (c) $\qquad X = -X_0$ then the node at midpoint



In general the distance h is

$$\frac{h}{X}=\frac{l-h}{X_0} \quad \Rightarrow \quad \frac{X}{X_0}=\frac{h}{l-h}$$



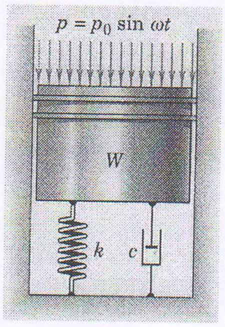
From equation (c)

$$h\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)=l-h$$

or
$$h=l\left(\frac{\omega_n}{\omega}\right)^2$$

Example

The 45-kg piston is supported by a spring of modulus $k=35\,\mathrm{kN/m}$. A dashpot of damping coefficient $c=1250\,\mathrm{N.s/m}$ acts in parallel with the spring. A fluctuating pressure $p(t)=4000\ \sin 30\,t$ in pascal acts on the piston, whose, whose area is $50\times10^{-3}\ m^2$. Determine the steady-state displacement as a function of time



Solution

The steady-state amplitude is X where

$$X=\frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \quad \cdots\cdots\cdots(a)$$

now, $\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{35\times10^3}{45}}=27.9\ \ rad/sec$ and $\zeta=\frac{c}{c_c}=\frac{c}{2m\omega_n}=\frac{1250}{2\times45\times27.9}=0.498$

$$F_0=pA=4000\times50\times10^{-3}=200\,N$$

$\omega=30\ rad/sec$ (given)

From Equation (a) $X=0.00528\,m=5.28\,mm$

$$\phi=\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}=\tan^{-1}\frac{2\times0.498\times\frac{30}{27.9}}{1-\left(\frac{30}{27.9}\right)^2}=1.716\quad rad$$

$$x(t)=5.28\ \ \sin(30t-1.716)$$

the transmitted force $=F_T=\sqrt{[kX]^2+[cX\omega]^2}$

$$F_T = kX\sqrt{1+\left[\frac{c\omega}{k}\right]^2} = kX\sqrt{1+\left[2\zeta\frac{\omega}{\omega_n}\right]^2} = 270.84\ N$$

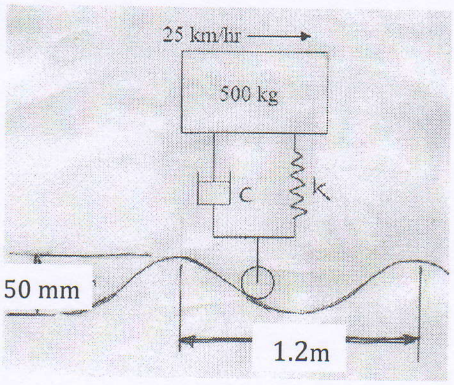
$$F_T = 35\times10^3\times0.00528\sqrt{1+[2\times0.498\times1.07527]^2} = 270.84\ N$$

HW

An electric motor is mounted on a steel table. The deflection of the table under the weight of the motor is observed to be 4 mm. the mass of the motor added to the effective mass of the table is 50 kg. The rotating parts of the motor have a mass of 10 kg and have an eccentricity of 12 mm. During free vibration, it is observed that the displacement of 40 mm is reduced to 2 mm in 1 second. Find the amplitude of motion if the operating speed of the motor is 1750 rpm. Assume the damping to be viscous.

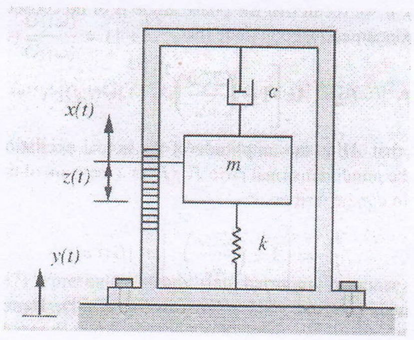
HW

Determine the amplitude of vertical vibration of the spring – mounted trailer as it travels at a velocity of 25 km/hr over the corduroy road whose contour may be expressed be a sine or cosine term. The mass of the trailer is 500 kg and that of the wheels alone may be negligible. During the loading 75 kg added to the load caused the trailer to sag 3 mm on its springs. Assume that the wheels are in contact with the road at all times and $\zeta = 0.5$. Also find the speed of the trailer which causes maximum vibration.



Vibration Measuring Instruments

Vibration measuring instruments used to indicate an output or response (displacement, velocity, acceleration) of machine.



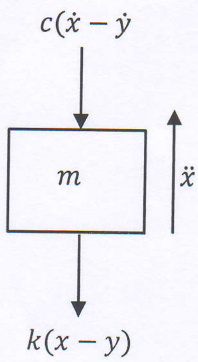
Let

$y(t)$:- motion of machine = motion of upper ends of spring and dashpot.

$x(t)$:- vertical displacement of suspending mass m.

$z(t)$:- motion of mass m relative to machine

If $x > y$



applying Newton's second law

$$-k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

after re arranging $\quad m\ddot{x} + k(x-y) + c(\dot{x}-\dot{y}) = 0 \quad \ldots\ldots\ldots\ldots (1)$

since $z(t) = x(t) - y(t) \implies \dot{z}(t) = \dot{x}(t) - \dot{y}(t)$ and $\ddot{z}(t) = \ddot{x}(t) - \ddot{y}(t)$

then, Equation (1) becomes to $\quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \quad \ldots\ldots\ldots\ldots (2)$

if $\quad y(t) = Y\cos\omega t \qquad \ddot{y}(t) = -Y\omega^2\cos\omega t$

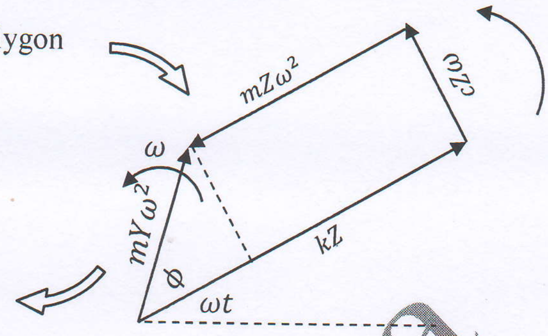
then,

$$z(t) = Z\cos(\omega t - \phi) \qquad \dot{z}(t) = -Z\omega\cos\left(\omega t - \phi + \frac{\pi}{2}\right) \qquad \ddot{z}(t) = -Z\omega^2\cos(\omega t - \phi)$$

From equation (2)

$$(kZ - mZ\omega^2)\cos(\omega t - \phi) - cZ\omega\cos\left(\omega t - \phi + \frac{\pi}{2}\right) = mY\omega^2\cos\omega t$$

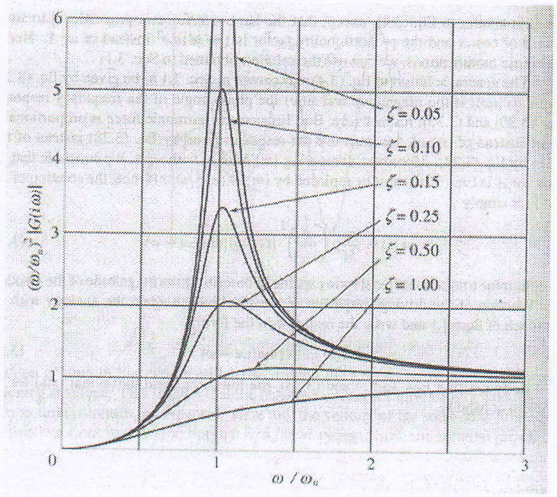
From this equation we can graph the following force polygon



then from this polygon we can extract that

$$\frac{Z}{Y} = \frac{m\omega^2}{\sqrt{[k-m\omega^2]^2+[c\omega]^2}} \times \frac{k}{k}$$

$$\boxed{\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}}$$ **Relative Motion Equation**



to find the maximum value $\frac{d}{d\omega}\left(\frac{Z}{Y}\right) = 0$

this yield $\frac{\omega}{\omega_n} = \frac{1}{\sqrt{1-2\zeta^2}}$ to the right of $\frac{\omega}{\omega_n} = 1$

Accelerometer

Accelerometer is an instruments that measures the acceleration of vibrating body.

if $y(t) = Y\cos\omega t$ $\quad \ddot{y}(t) = -Y\omega^2\cos\omega t$ .... (1)

then, $\quad z(t) = Z\cos(\omega t - \phi)$ .... (2)

from relative motion equation, let $r = \frac{\omega}{\omega_n}$

$$Z = \frac{r^2 Y}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$$

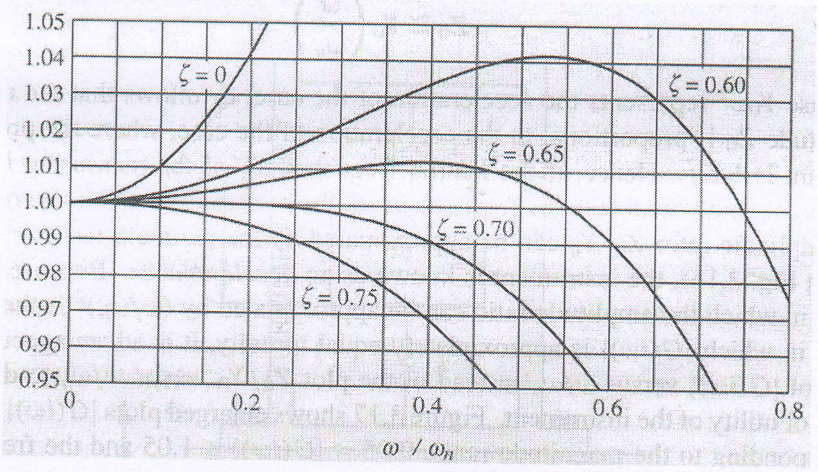
so from Equation (2) $\quad z(t) = \frac{r^2 Y}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}\cos(\omega t - \phi) \times \frac{-\omega_n^2}{-\omega_n^2}$

$$-z(t)\omega_n^2 = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}\underbrace{[-Y\omega^2\cos(\omega t - \phi)]}_{\ddot{y}(t)}$$

or $\quad -z(t)\omega_n^2 = \ddot{y}(t)$ $\quad$ if $\quad \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} = 1$

the best value of ζ is 0.707

and $\quad 0 < \frac{\omega}{\omega_n} < 0.6$



Example

Find the spring stiffness and the damping coefficient of an accelerometer that can be measured vibrations in the range of 0 to 40 Hz with maximum error of 0.5% if $m = 0.05$ kg.

Solution

for maximum error of $0.5\% = 0.005$ $\quad$ so $\quad 1 - 0.005 = 0.995$

$$0.995 = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} \quad \Rightarrow \quad r^4 - 2r^2(1-2\zeta^2) - 0.010078 = 0.995$$

for accelerometer the best value of ζ is 0.707

$$r^4 - 0.04r^2 - 0.010078 = 0$$

which gives the positive value of $\quad r^2 = 0.12235 \quad \Rightarrow \quad r = 0.349785$

since $\quad \omega = 2\pi f = 2\pi \times 40 = 251.328 \ rad/sec$ [The maximum value of forcing frequency]

$$r = \frac{\omega}{\omega_n} \quad \Rightarrow \quad \omega_n = \frac{251.328}{0.349785} = 718.52 \ rad/sec$$

$$k = m\omega_n^2 \qquad k = (0.05) \times (718.52)^2 = 25813.57 \ N/m$$

and $\quad c = 2m\omega_n\zeta = 2 \times 0.05 \times 718.52 \times 0.707 = 508 \ \frac{N.s}{m}$